PHYS170: Mechanics I
Term 2015WT2, Section 201; Fatemeh Renani
Engineering Mechanics: Statics and Dynamics, 12th Edition; R. C. Hibbeler

# 1 General Principles

## 1.1 Mechanics

**mechanics**: the study of bodies subject to forces
**statics**: the study of the mechanics of bodies in equilibrium

## 1.2 Fundamental Concepts

**length** ($l$): a measure of the size of a body
**time** ($t$): a measure of the succession of events
**force** ($\mathbf{F}$): an interaction with a body
**mass** ($m$): a measure of the resistance of a body to forces
**particle**: a body with mass but no size
**rigid body**: a body whose constituent particles remain in a fixed distance from each other
**deformable body**: a body whose constituent particles can move relative to each other
**concentrated force**: a force which is applied to a single point on a body
**Newton's first law**: a particle at rest or moving with constant velocity tends to remain so
**Newton's second law**: a particle subjected to an unbalanced force accelerates with the force
**Newton's third law**: two particles always exert equal and opposite forces on each other
Newton's laws apply only when motion is measured from a reference frame with no acceleration.
**Newton's law of gravitational attraction**: the gravitational force between two particles is proportional to their masses and inversely proportional to the square of the length between them
**weight** ($\mathbf{W}$): the force of gravity on a particle near the surface of the earth
**acceleration due to gravity** ($\mathbf{g}$): the acceleration due to the force of gravity on a particle at a latitude of 45 degrees at sea level on the earth

$$\begin{aligned} \mathbf{W} &= m\mathbf{g} \\ g &= 9.81\,\mathrm{m}\cdot\mathrm{s}^{-2} = 32.2\,\mathrm{ft}\cdot\mathrm{s}^{-2} \end{aligned} \tag{1}$$

## 1.3 Units of Measurement

**SI units**: the International System of units
**metre** (m): the SI fundamental unit of length
**second** (s): the SI and FPS fundamental unit of time
**kilogram** (kg): the SI fundamental unit of mass
**Newton** (N): the Si derived unit of force, $1\,\mathrm{N} = 1\,\mathrm{kg}\cdot\mathrm{m}\cdot\mathrm{s}^{-2}$
**FPS units**: the United States Customary system of units
**foot** (ft): the FPS fundamental unit of length, $1\,\mathrm{ft} = 0.3048\,\mathrm{m}$
**pound** (lb): the FPS fundamental unit of force, $1\,\mathrm{lb} = 4.448\,\mathrm{N}$
**slug** (slug): the FPS derived unit of mass, $1\,\mathrm{slug} = 14.59\,\mathrm{kg}$
**inch** (in): another FPS unit of length, $1\,\mathrm{in} = 1/12\,\mathrm{ft}$
**mile** (mi): another FPS unit of length, $1\,\mathrm{mi} = 5280\,\mathrm{ft}$
**kilo-pound** (kip): another FPS unit of force, $1\,\mathrm{kip} = 1000\,\mathrm{lb}$
**ton** (ton): another FPS unit of force, $1\,\mathrm{ton} = 2000\,\mathrm{lb}$

### 1.4 The International System of Units

SI units have prefixes to indicate the order of magnitude. Avoid prefixes in denominators.
**giga** (G): the SI prefix for multiplication by $10^9$
**mega** (M): the SI prefix for multiplication by $10^6$
**kilo** (k): the SI prefix for multiplication by $10^3$
**milli** (m): the SI prefix for multiplication by $10^{-3}$
**micro** ($\mu$): the SI prefix for multiplication by $10^{-6}$
**nano** (n): the SI prefix for multiplication by $10^{-9}$

### 1.5 Numerical Calculations

**dimensional homogeneity**: the property of an equation whose additive terms all have the same units; all equations that describe physical processes must be dimensionally homogeneous
**significant figures**: the number of digits whose values are known within experimental accuracy
**engineering notation**: the representation of numbers in multiples of $10^3$
Always use engineering notation, in order to properly specify the number of significant figures. Round numbers that end above 5 up and numbers that end below 5 down. Round numbers that end exactly on 5 up if the previous digit is odd and down if the previous digit is even. Numbers are usually rounded to three significant figures. Do not round any intermediate calculations.

### 1.6 General Procedure for Analysis

Objects are usually modelled as particles or rigid bodies, and forces are usually modelled as concentrated forces, in order to simplify analysis. Always analyze in a logical and orderly manner.

## 2 Force Vectors

### 2.1 Scalars and Vectors

**scalars**: physical quantities that do not have direction
**vectors**: physical quantities that have direction and positive magnitude
Scalars can be positive or negative. The sense of a vector is its direction forwards or backwards. Vectors are represented by arrows, whose length is their magnitude and whose direction is their direction. Vectors $\mathbf{v}$ are bolded and their corresponding signed magnitudes $v$ are not.

### 2.2 Vector Operations

**resultant vector**: the result of a vector operation
To multiply a vector by a positive scalar, create a resultant vector in the same direction and multiply its magnitude by the scalar. To multiply a vector by a negative scalar, create a resultant vector in the opposite direction and multiply its magnitude by the absolute value of the scalar. To add two vectors, place the tip of the first on the tail of the second, and create a resultant vector from first tail to second tip. To subtract two vectors, add the first to $-1$ times the second.

### 2.3 Vector Addition of Forces

A triangle with sides $A$, $B$, $C$ and opposite angles $a$, $b$, $c$ obeys the sine and cosine laws:

$$\frac{A}{\sin a} = \frac{B}{\sin b} = \frac{C}{\sin c} \qquad C^2 = A^2 + B^2 - 2AB\cos c \tag{2}$$

## 2.4 Addition of a System of Coplanar Forces

**components**: vectors that add together to become the original vector
**rectangular components**: components along the Cartesian axes
**unit vector** (**u**): a vector whose magnitude is one and dimensionless
**Cartesian unit vectors**: unit vectors along the Cartesian axes
The Cartesian unit vectors are **i** along the $x$ axis, **j** along the $y$ axis, and **k** along the $z$ axis.
If the rectangular components of force **F** are $\mathbf{F}_x$ and $\mathbf{F}_y$, and the angle between **F** and the positive $x$ axis is $\theta$:

$$\begin{aligned}\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y = F_x\,\mathbf{i} + F_y\,\mathbf{j} \\ F = \sqrt{F_x^2 + F_y^2}\end{aligned} \tag{3}$$

$$\theta = \tan^{-1}\left(\frac{F_y}{F_x}\right) \tag{4}$$

$$F_x = F\cos\theta \qquad F_y = F\sin\theta \tag{5}$$

## 2.5 Cartesian Vectors

**right-handed coordinate system**: the conventional Cartesian coordinate system, such that if the positive $x$ axis points with your right index and the positive $y$ axis points with your right middle, then the positive $z$ axis points with your right thumb
**coordinate direction angles**: the angles between a vector and the Cartesian axes
**direction cosines**: the cosines of the coordinate direction angles
The coordinate direction angles are $\alpha$ for the $x$ axis, $\beta$ for the $y$ axis, and $\gamma$ for the $z$ axis.
The corresponding direction cosines are $\cos\alpha$, $\cos\beta$, and $\cos\gamma$.
If the rectangular components of force **F** are $\mathbf{F}_x$, $\mathbf{F}_y$, $\mathbf{F}_z$, the angle between **F** and the positive $z$ axis is $\phi$, and the angle between the projection of **F** on the $xy$ plane and the positive $x$ axis is $\theta$:

$$\begin{aligned}\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y + \mathbf{F}_z = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k} \\ F = \sqrt{F_x^2 + F_y^2 + F_z^2}\end{aligned} \tag{6}$$

$$\theta = \tan^{-1}\left(\frac{F_y}{F_x}\right) \qquad \phi = \cos^{-1}\left(\frac{F_z}{F}\right) \tag{7}$$

$$F_x = F\sin\phi\cos\theta \qquad F_y = F\sin\phi\sin\theta \qquad F_z = F\cos\phi \tag{8}$$

$$F_x = F\cos\alpha \qquad F_y = F\cos\beta \qquad F_z = F\cos\gamma \tag{9}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{10}$$

## 2.6 Addition of Cartesian Vectors

**resultant force** ($\mathbf{F}_R$): the vector sum of all forces acting on a body

$$\mathbf{F}_R = \Sigma\,\mathbf{F} = (\Sigma\,F_x)\,\mathbf{i} + (\Sigma\,F_y)\,\mathbf{j} + (\Sigma\,F_z)\,\mathbf{k} \tag{11}$$

## 2.7 Position Vectors

**origin** ($O$): the reference point in a coordinate system
**coordinates**: the location of a point in a coordinate system relative to the origin
If point $A$ is $x_A$ away from $O$ along the $x$ axis, $y_A$ along the $y$ axis, and $z_A$ along the $z$ axis:

$$A(x_A, y_A, z_A) \tag{12}$$

**position vector** ($\mathbf{r}$): the vector from the origin to the location of a point

$$\mathbf{r}_A = x_A\mathbf{i} + y_A\mathbf{j} + z_A\mathbf{k} \tag{13}$$

The position vector of point $A$ relative to point $B$ is:

$$\mathbf{r}_{BA} = \mathbf{r}_B - \mathbf{r}_A \tag{14}$$

## 2.8 Force Vector Directed Along a Line

If a force is directed along a line through two points $A$ and $B$:

$$\mathbf{F} = F\mathbf{u}_{AB} = F\left(\frac{1}{r_{BA}}\mathbf{r}_{BA}\right) \tag{15}$$

## 2.9 Dot Product

**dot product** ($\cdot$): the operator that outputs a scalar related to the angle of the input vectors
If the angle between vectors $\mathbf{A}$ and $\mathbf{B}$ is $\theta$:

$$\begin{aligned}\mathbf{A} \cdot \mathbf{B} &= AB\cos\theta \\ &= A_xB_x + A_yB_y + A_zB_z\end{aligned} \tag{16}$$

The components parallel $\mathbf{A}_{\parallel}$ and perpendicular $\mathbf{A}_{\perp}$ to the unit vector $\mathbf{u}$ are:

$$\begin{aligned}\mathbf{A}_{\parallel} &= (\mathbf{A} \cdot \mathbf{u})\,\mathbf{u} \\ \mathbf{A}_{\perp} &= \mathbf{A} - \mathbf{A}_{\parallel}\end{aligned} \tag{17}$$

# 3 Equilibrium of a Particle

## 3.1 Condition for the Equilibrium of a Particle

**equilibrium**: the state of a particle on which there is no resultant force
**static equilibrium**: the state of a particle in equilibrium that does not move
The necessary and sufficient condition for a particle to be in equilibrium is:

$$\mathbf{F}_R = \mathbf{0} \tag{18}$$

### 3.2 The Free-Body Diagram

**free-body diagram**: a diagram that shows a body with all the external forces that act on it
The internal forces in a body cancel out, so that only the external forces affect its acceleration.
**spring constant** or stiffness ($k$): the proportionality constant between the change in length of and the force on a spring
The force exerted on an ideal linearly elastic spring only exists if the spring is compressed.
Its magnitude is proportional to the change between the current length $l$ and the equilibrium length $l_0$ of the spring, and its direction is along the spring, away from its equilibrium position.

$$F = k\,(l - l_0) \tag{19}$$

**tension** ($\mathbf{T}$): the force that acts at every point along a rope
The tension on an ideal rope only exists if the rope is not slack. Its magnitude is constant at every point, and its direction is along the rope. Ideal ropes have no weight and cannot stretch.

### 3.3 Coplanar Force Systems

Use signs to specify the sense. If the result is negative, then the sense is opposite to that assumed.
The necessary and sufficient conditions for a particle to be in equilibrium are:

$$\Sigma\,F_x = 0 \qquad \Sigma\,F_y = 0 \tag{20}$$

### 3.4 Three-Dimensional Force Systems

The necessary and sufficient conditions for a particle to be in equilibrium are:

$$\Sigma\,F_x = 0 \qquad \Sigma\,F_y = 0 \qquad \Sigma\,F_z = 0 \tag{21}$$

## 4 Force System Resultants

### 4.1 Moment of a Force—Scalar Formulation

**moment** or torque ($\mathbf{M}_O$): the tendency to rotate about point $O$ due to an applied force
**moment arm** ($d$): the distance between the axis of rotation and the line of action of the force
**resultant moment** ($(\mathbf{M}_R)_O$): the vector sum of all moments acting on a body about point $O$
Positive moments try to rotate bodies counterclockwise. the resultant moment is a signed sum:

$$\begin{gathered} M_O = Fd \\ (M_R)_O = \Sigma\,(\pm Fd) \end{gathered} \tag{22}$$

**free vector**: the vector of a physical quantity that can act at any point on a body
**sliding vector**: the vector of a physical quantity that can act at any point along its direction
Moments are neither free nor sliding vectors; they depend on the point about which they act.

## 4.2 Cross Product

**cross product** ($\times$): the operator that outputs a vector perpendicular to both input vectors
**right hand rule**: the procedure to find the direction of the cross product: if the first operand points with your right index and the second operand points with your right middle, then the cross product points with your right thumb; and if there is a tendency to rotate along the direction that your right hand fingers curl, then the moment points with your right thumb
If the angle between $\mathbf{A}$ and $\mathbf{B}$ is $\theta$, and $\mathbf{u}$ is the unit vector of $\mathbf{A} \times \mathbf{B}$ from the right hand rule:

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= (AB \sin\theta)\,\mathbf{u} \\ &= (A_y B_z - A_z B_y)\,\mathbf{i} + (A_z B_x - A_x B_z)\,\mathbf{j} + (A_x B_y - A_y B_x)\,\mathbf{k} \end{aligned} \tag{23}$$

## 4.3 Moment of a Force—Vector Formulation

**principle of transmissibility**: the application of a force at any point along its line of action creates the same moment about any point, so all forces are sliding vectors
If $\mathbf{r}$ is a position vector from $O$ to any point on the line of action of $\mathbf{F}$:

$$\begin{aligned} \mathbf{M}_O &= \mathbf{r} \times \mathbf{F} \\ &= (r_y F_z - r_z F_y)\,\mathbf{i} + (r_z F_x - r_x F_z)\,\mathbf{j} + (r_x F_y - r_y F_x)\,\mathbf{k} \\ (\mathbf{M}_R)_O &= \Sigma\ (\mathbf{r} \times \mathbf{F}) \end{aligned} \tag{24}$$

## 4.4 Principle of Moments

**principle of moments** or Varignon's theorem: the moment of a force about a point is equal to the sum of the moments of the components of the force about the point

## 4.5 Moment of a Force about a Specified Axis

The moment $\mathbf{M}_a$ about an axis with unit vector $\mathbf{u}_a$ and moment arm $d_a$ is the projection of the moment $\mathbf{M}_O$ about any point $O$ on the axis onto the axis:

$$\begin{aligned} M_a = F d_a &= \mathbf{u}_a \cdot \mathbf{M}_O \\ \mathbf{M}_a &= (\mathbf{u}_a \cdot (\mathbf{r} \times \mathbf{F}))\,\mathbf{u}_a \end{aligned} \tag{25}$$

## 4.6 Moment of a Couple

**couple**: two parallel forces that have the same magnitude and opposite directions
**couple moment** ($\mathbf{M}$): the moment produced by a couple
**equivalent couples**: two couples that produce exactly the same moment
A couple moment is the same about any point, so all couple moments are free vectors.
If points $A$ and $B$ a distance $d$ apart lie along the lines of action of $-\mathbf{F}$ and $\mathbf{F}$ respectively:

$$\begin{aligned} M &= Fd \\ \mathbf{M} &= \mathbf{r}_{BA} \times \mathbf{F} \\ \mathbf{M}_R &= \Sigma\ (\mathbf{r}_{BA} \times \mathbf{F}) \end{aligned} \tag{26}$$

### 4.7 Simplification of a Force and Couple System

**equivalent system**: a single force acting at a given point and a single couple that produce exactly the same resultant force and resultant moment as a given system of forces
For a given system of couple moments $\mathbf{M}$ and forces $\mathbf{F}$ whose lines of action are $\mathbf{r}$ away from the given point $O$, an equivalent system with force $\mathbf{F}_R$ and couple moment $(\mathbf{M}_R)_O$ can be made by:

$$\begin{aligned} \mathbf{F}_R &= \Sigma\,\mathbf{F} \\ (\mathbf{M}_R)_O &= \Sigma\,\mathbf{M} + \Sigma\,(\mathbf{r} \times \mathbf{F}) \end{aligned} \tag{27}$$

### 4.8 Further Simplification of a Force and Couple System

**concurrent force system**: a system of forces whose lines of action all intersect at a single point
**coplanar force system**: a system of forces whose lines of action all lie in a single plane
**parallel force system**: a system of forces whose lines of action are all parallel
**wrench**: an equivalent system whose force and moment are parallel
A concurrent force system can be simplified to a single force acting on the point of intersection.
A coplanar force system can be simplified to a single force by moving the resultant force that acts on point $O$ to a distance $d = (\mathbf{M}_R)_O/\mathbf{F}_R$ away so that it produces the resultant moment.
A parallel force system can be simplified to a single force in exactly the same way.
Any force system can be simplified to a wrench by moving the resultant force so that it produces the perpendicular component of the resultant moment.

## 5 Equilibrium of a Rigid Body

### 5.1 Conditions for Rigid-Body Equilibrium

The necessary and sufficient conditions for a rigid body to be in equilibrium are:

$$\begin{aligned} \mathbf{F}_R &= \mathbf{0} \\ (\mathbf{M}_R)_O &= \mathbf{0} \end{aligned} \tag{28}$$

### 5.2 Free-Body Diagrams

If a support prevents a translation in some direction, then the support exerts a reactive force on the body in that direction, and if a support prevents a rotation, then the support exerts a couple moment on the body. Possible ideal smooth supports in a coplanar force system include:

- cable: one reactive force that acts away from the member along the direction of the cable;
- link: one reactive force that acts along the direction of the link;
- roller, rocker, or smooth surface: one reactive force that acts toward the member perpendicular to the surface at the point of contact;
- roller or pin in confined slot: one reactive force that acts perpendicular to the slot;
- member pin-connected to collar: one reactive force that acts perpendicular to the rod;
- member fixed-connected to collar: one reactive force that acts perpendicular to the rod of the collar and one reactive couple moment;
- pin or hinge: only the moment that acts through the pin or hinge is unconstrained;
- fixed support: everything is constrained.

The weight of a body always acts downwards at its centre of gravity, which is the one point by which the body can be balanced. This is at the geometric centre of bodies with constant density.

### 5.3 Equations of Equilibrium

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma M_O = 0 \tag{29}$$

### 5.4 Two- and Three-Force Members

**two-force member**: a body on which there are only two external forces
**three-force member**: a body on which there are only three external forces
For any two-force member to be in equilibrium, the two forces must have the same magnitude and act in opposite directions along the line joining their points of application. For any three-force member to be in equilibrium, the three forces must form a concurrent or parallel force system.

### 5.5 Free-Body Diagrams

Possible ideal smooth supports in a general force system include:

- ball and socket: all forces are constrained;
- journal bearing: two reactive forces and two moments, all acting perpendicular to the shaft;
- journal bearing with square shaft: only the force that acts along the shaft is unconstrained;
- thrust bearing: only the moment that acts along the shaft is unconstrained.

When bearings, pins, and hinges are properly used in conjunction, they do not exert any reactive moments, and their reactive forces alone are adequate to maintain equilibrium.

### 5.6 Equations of Equilibrium

$$\begin{matrix} \Sigma F_x = 0 & \Sigma F_y = 0 & \Sigma F_z = 0 \\ \Sigma M_x = 0 & \Sigma M_y = 0 & \Sigma M_z = 0 \end{matrix} \tag{30}$$

### 5.7 Constraints and Statical Determinacy

**statically indeterminate**: the condition of a body that has more supports than needed to maintain equilibrium, so that its reactive forces cannot be solved with static equilibrium equations
If the lines of action of all its reactive forces are parallel or intersect in a common axis, then a body is improperly constrained and may not remain in equilibrium. If there is not enough reactive forces, then a body is partially constrained and cannot remain in equilibrium.

## 8 Friction

### 8.1 Characteristics of Dry Friction

**normal force** (**N**): the component of the reactive force that acts perpendicular to a surface
**frictional force** (**F**): the component of the reactive force that acts along a deformable surface
The normal force ensures that contacting surfaces don't pass through each other, and the frictional force opposes the direction of possible or existing motion. On a rigid body, the normal force and frictional force act on a point a distance $x$ away from the perpendicular component of the external force on the body, so that moment is zero. If the only external forces on a rigid box are its weight **W** and a horizontal force **P** applied at a height $h$ from the surface, then:

$$x = Ph/W \tag{31}$$

**static frictional force** ($\mathbf{F}_s$): the frictional force before slipping occurs
**limiting static frictional force** ($F_s$): the maximum magnitude of $\mathbf{F}_s$ before slipping occurs
**coefficient of static friction** ($\mu_s$): the proportionality constant between $\mathbf{F}_s$ and $\mathbf{N}$
**angle of static friction** ($\phi_s$): the angle between $\mathbf{N}$ and $\mathbf{F}_s + \mathbf{N}$
This is also the angle of the maximum incline on which a box will not slip.
If $|x|$ is more than half the width, the box will tip. If $P > F_s$, the box will slip.
**kinetic frictional force** ($\mathbf{F}_k$): the frictional force while slipping occurs
**coefficient of kinetic friction** ($\mu_k$): the proportionality constant between $\mathbf{F}_k$ and $\mathbf{N}$
**angle of kinetic friction** ($\phi_k$): the angle between $\mathbf{N}$ and $\mathbf{F}_k + \mathbf{N}$

$$\begin{aligned} F_s &= \mu_s N & \phi_s &= \tan^{-1}\mu_s \\ F_k &= \mu_k N \approx 0.75 F_s & \phi_k &= \tan^{-1}\mu_k > \phi_s \end{aligned} \tag{32}$$

### 8.2 Problems Involving Dry Friction

On a free-body diagram, frictional forces must have the correct sense, and their magnitudes must not be related to the normal force unless that is given. Instead, solve for their magnitudes.
If the number of unknowns is equal to the number of equilibrium equations, then determine the frictional forces and check that $F \leq \mu_s N$ for all of them. If the number of unknowns is equal to the number of equilibrium equations and frictional equations, then apply $F = \mu_s N$ or $F = \mu_k N$ for the frictional forces to determine the other forces. Otherwise, apply $F = \mu_s N$ or $F = \mu_k N$ for all possible choices of the frictional forces to determine the other forces.

### 8.3 Wedges

**wedge**: an inclined plane
**self-locking wedge**: a wedge in equilibrium without any external forces
Wedges usually have negligible weight. Otherwise, they are the same as any other body.

## 12 Kinematics of a Particle

### 12.1 Introduction

**dynamics**: the study of the mechanics of accelerating bodies
**kinematics**: the study of the geometry behind dynamics

### 12.2 Rectilinear Kinematics: Continuous Motion

**rectilinear kinematics**: the study of kinematics in a straight line
Any rigid body can be modelled as a non-rotating particle at its centre of mass. The kinematics of a particle is completely defined by its instantaneous position, velocity, and acceleration.
The following quantities are scalars, for when a particle's path lies along a single coordinate axis.
**distance** ($s_T$): the total length of path over which a particle travels during some time
**position** ($s$): the signed distance from the fixed origin $O$ to the location of a particle
**displacement** ($\Delta s$): the change in the position of a particle after some time
**average velocity** ($v_{avg}$): the displacement of a particle during some time, divided by that time
**instantaneous velocity** ($v$): the rate of change of the position of a particle
**average speed** ($(v_{sp})_{avg}$): the distance of a particle during some time, divided by that time
**instantaneous speed** ($v_{sp}$): the magnitude of the instantaneous velocity of a particle

**average acceleration** ($a_{avg}$): the change in the instantaneous velocity of a particle after some time, divided by that time
**instantaneous acceleration** ($a$): the rate of change of the instantaneous velocity of a particle
A body is accelerating if its instantaneous speed is increasing and decelerating if it is decreasing.
Quantities $x$ have a subscript zero $x_0$ when $t = 0$, and changes in $x$ have a delta $\Delta x = x - x_0$:

$$\begin{aligned} \Delta s &= s - s_0 \\ v_{avg} &= \frac{\Delta s}{\Delta t} & v &= \frac{\mathrm{d}s}{\mathrm{d}t} \\ (v_{sp})_{avg} &= \frac{s_T}{\Delta t} & v_{sp} &= |v| \\ a_{avg} &= \frac{\Delta v}{\Delta t} & a &= \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} \\ a\ \mathrm{d}s &= v\ \mathrm{d}v \end{aligned} \tag{33}$$

When a particle's acceleration is a constant $a = a_c$, these equations can be integrated to obtain:

$$\begin{aligned} v &= v_0 + a_c t \\ s &= s_0 + v_0 t + \frac{a_c}{2} t^2 \\ v^2 &= v_0^2 + 2a_c (s - s_0) \end{aligned} \tag{34}$$

## 12.3 Rectilinear Kinematics: Erratic Motion

The position of particles with erratic motion cannot be described by a single continuous mathematical function, so it is useful to look at graphs instead.
The $s$-$t$ graph has a slope of $v$. The $v$-$t$ graph has a slope of $a$, and an area of $\Delta s$. The $a$-$t$ graph has an area of $\Delta v$. The $v$-$s$ graph has a slope of $a/v$. The $a$-$s$ graph has an area of $\left(v^2 - v_0^2\right)/2$.

## 12.4 General Curvilinear Motion

**curvilinear motion**: the motion of a particle along a curved path
**arclength** ($s$): the distance from the fixed origin $O$ to the location of a particle along the path
**position** ($\mathbf{r}$): the vector from the fixed origin $O$ to the location of a particle
**velocity** ($\mathbf{v}$): the rate of change of the position of a particle
**speed** ($v$): the magnitude of the velocity of a particle
**acceleration** ($\mathbf{a}$): the rate of change of the velocity of a particle
If a particle moves along the path defined by the arclength function $s(t)$, then its instantaneous position is a vector, and its displacement and average and instantaneous velocity, speed, and acceleration are vector functions that depend on its vector position $\mathbf{r}$, not its scalar position $s$:

$$\begin{aligned} \Delta \mathbf{r} &= \mathbf{r} - \mathbf{r}_0 \\ \mathbf{v}_{avg} &= \frac{\Delta \mathbf{r}}{\Delta t} & \mathbf{v} &= \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \\ v_{avg} &= \frac{\Delta s}{\Delta t} & v &= \|\mathbf{v}\| = \frac{\mathrm{d}s}{\mathrm{d}t} \\ \mathbf{a}_{avg} &= \frac{\Delta \mathbf{v}}{\Delta t} & \mathbf{a} &= \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} \end{aligned} \tag{35}$$

**hodograph**: a space curve defined by the velocity function $\mathbf{v}(t)$
The directions of velocity and acceleration are tangent to the path and hodograph, respectively.

## 12.5 Curvilinear Motion: Rectangular Components

It is possible to express all the vectors in the previous subsection with Cartesian coordinates in three dimensions, as long as the reference frame $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ is fixed. If $\dot{x}$, $\dot{y}$, $\dot{z}$ are the first and $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ are the second time derivatives of $x = x(t)$, $y = y(t)$, $z = z(t)$, and $s = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then:

$$\begin{aligned} \mathbf{r} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\ \mathbf{v} &= \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \\ \mathbf{a} &= \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k} \end{aligned} \tag{36}$$

## 12.6 Motion of a Projectile

If air resistance is absent so the only force is weight, a projectile moves with $\mathbf{a}_x = 0$ and $\mathbf{a}_y = \mathbf{g}$.

## 12.7 Curvilinear Motion: Normal and Tangential Components

**unit tangent vector** ($\mathbf{u}_t$): the rate of change of arclength with respect to time
**unit normal vector** ($\mathbf{u}_n$): the rate of change of the unit tangent vector with respect to time
**unit binormal vector** ($\mathbf{u}_b$): the cross product of the unit tangent and normal vectors

$$\begin{aligned} \mathbf{u}_t &= \frac{1}{\|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\|}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \\ \mathbf{u}_n &= \frac{\mathrm{d}\mathbf{u}_t}{\mathrm{d}t} \\ \mathbf{u}_b &= \mathbf{u}_t \times \mathbf{u}_n \end{aligned} \tag{37}$$

**radius of curvature** ($\rho$): the radius of the circle whose arc describes the path at an instant
**osculating plane**: the plane that contains the unit tangent and principal unit normal vectors
**centripetal force**: the force that causes an acceleration in the direction of the normal vector
The vectors in the past subsections can be expressed with TNB coordinates on an osculating plane, whose fixed origin coincides with the position of the particle and whose axes are $\mathbf{u}_t$ and $\mathbf{u}_n$. If $a_t$ and $a_n$ are the accelerations in the direction of the tangent and normal vectors, respectively:

$$\dot{\mathbf{u}}_t = \left(v^2/\rho\right)\mathbf{u}_n \tag{38}$$

$$\begin{aligned} \mathbf{v} &= v\mathbf{u}_t \\ \mathbf{a} = \dot{\mathbf{v}} &= a_t\mathbf{u}_t + a_n\mathbf{u}_n \end{aligned} \tag{39}$$

$$a_t = \dot{v} \qquad a_n = v^2/\rho \tag{40}$$

## 12.8 Curvilinear Motion: Cylindrical Components

**radial coordinate** ($r$): the distance from the fixed origin $O$ to the position of a particle
**transverse coordinate** ($\theta$): the counterclockwise angle between a fixed reference line lying on a fixed reference plane and the position of a particle
**azimuthal coordinate** ($z$): the signed distance from the fixed plane to the position of a particle
**angular velocity** ($\dot{\theta}$): the rate of change of the transverse coordinate of a particle
**angular acceleration** ($\ddot{\theta}$): the rate of change of the angular velocity of a particle

**cylindrical coordinates**: a coordinate system with a fixed origin $O$ and whose axes are $\mathbf{u}_r$, $\mathbf{u}_\theta$, and $\mathbf{u}_z$, the positive directions of the radial, transverse, and azimuthal coordinates respectively
**polar coordinates**: a cylindrical coordinate system constrained to one plane, so that $\mathbf{u}_z = 0$
The vectors in the past subsections can be expressed with cylindrical coordinates in three or polar coordinates in two dimensions, as long as the reference frame $u_r$, $u_\theta$, $u_z$ is fixed:

$$\dot{\mathbf{u}}_r = \dot{\theta}\mathbf{u}_\theta \qquad \dot{\mathbf{u}}_\theta = -\dot{\theta}\mathbf{u}_r \tag{41}$$

$$\begin{aligned} \mathbf{r} &= r\mathbf{u}_r + z\mathbf{u}_z \\ \mathbf{v} = \dot{\mathbf{r}} &= v_r\mathbf{u}_r + v_\theta\mathbf{u}_\theta + v_z\mathbf{u}_z \\ \mathbf{a} = \dot{\mathbf{v}} &= a_r\mathbf{u}_r + a_\theta\mathbf{a}_\theta + a_z\mathbf{u}_z \end{aligned} \tag{42}$$

$$\begin{aligned} v_r &= \dot{r} & v_\theta &= r\dot{\theta} & v_z &= \dot{z} \\ a_r &= \ddot{r} - r\dot{\theta}^2 & a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta} & a_z &= \ddot{z} \end{aligned} \tag{43}$$

### 12.9 Absolute Dependent Motion Analysis of Two Particles

**datum line**: a fixed line from which to measure position coordinates
Specify the coordinates of dependent particles as distances measured from a datum line along the direction of motion of each. Add the coordinates up to equal some constant, then differentiate.

### 12.10 Relative Motion of Two Particles Using Translating Axes

**relative position** ($\mathbf{r}_{BA}$): the position $\mathbf{r}_B$ of a particle measured from the position $\mathbf{r}_A$ of another
If particle $B$ is observed from the non-rotating reference frame whose origin is at particle $A$:

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{BA} \qquad \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{BA} \qquad \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{BA} \tag{44}$$

## 13 Kinetics of a Particle: Force and Acceleration

### 13.1 Newton's Second Law of Motion

**kinetics**: the study of the forces behind dynamics
When speeds are much slower than the speed of light and sizes are much larger than the size of atoms, Newton's Second Law of Motion applies:

$$\mathbf{F} = m\mathbf{a} = \frac{\mathrm{d}m\mathbf{v}}{\mathrm{d}t} \tag{45}$$

### 13.2 The Equation of Motion

**kinetic diagram**: a diagram that shows the resultant force of a free body diagram
**inertial reference frame**: a non-rotating reference frame that translates at a constant velocity
Newton's Second Law is only valid in an inertial reference frame, and relates the resultant force on a particle to its acceleration:

$$\mathbf{F}_R = m\mathbf{a} \tag{46}$$

### 13.3 Equation of Motion for a System of Particles

**centre of mass** ($\mathbf{r}_G$): the one point in a body such that all external forces whose lines of action pass through it create no rotation of the body
In a system of $i$ particles with total mass $m$, where each particle has mass $m_i$ and is subjected to resultant internal forces $\mathbf{f}_i$ and resultant external forces $\mathbf{F}_i$, all internal forces cancel out:

$$\begin{aligned} m\mathbf{r}_G &= \Sigma\,(m_i\mathbf{r}_i) \\ \Sigma\,\mathbf{F}_i + \Sigma\,\mathbf{f}_i &= \Sigma\,(m_i\mathbf{a}_i) \\ \mathbf{F}_R &= m\mathbf{a}_G \end{aligned} \tag{47}$$

### 13.4 Equations of Motion: Rectangular Coordinates

The equations of motion can be expressed with Cartesian coordinates in three dimensions:

$$\Sigma\,F_x = m\ddot{x} \qquad \Sigma\,F_y = m\ddot{y} \qquad \Sigma\,F_z = m\ddot{z} \tag{48}$$

### 13.5 Equations of Motion: Normal and Tangential Coordinates

The equations of motion can be expressed with TNB coordinates on the osculating plane:

$$\Sigma\,F_t = m\dot{v} \qquad \Sigma\,F_n = m\frac{v^2}{\rho} \qquad \Sigma\,F_b = 0 \tag{49}$$

### 13.6 Equations of Motion: Cylindrical Coordinates

The equations of motion can be expressed with cylindrical coordinates in three dimensions or polar coordinates in two dimensions:

$$\Sigma\,F_r = m\left(\ddot{r} - r\dot{\theta}^2\right) \qquad \Sigma\,F_\theta = m\left(2\dot{\theta}\dot{r} + r\ddot{\theta}\right) \qquad \Sigma\,F_z = m\ddot{z} \tag{50}$$

If $\psi$ is the counterclockwise angle from the unit radial vector to the unit tangent vector:

$$\tan\psi = r\frac{\mathrm{d}\theta}{\mathrm{d}r} \tag{51}$$

## 14 Kinetics of a Particle: Work and Energy

### 14.1 The Work of a Force

**work** ($U$): the product of the force on a particle and the component of the particle's displacement in the direction of the force
**energy**: the capacity for a particle to do work
**Joule** (J): the SI unit of work and energy, $1\,\mathrm{J} = 1\,\mathrm{N}\cdot\mathrm{m}$

$$\begin{aligned} \mathrm{d}U &= \mathbf{F}\cdot\mathrm{d}\mathbf{r} & & \\ U &= \mathbf{F}\cdot\Delta\mathbf{r} & &\text{constant } \mathbf{F} \text{ along straight line} \\ &= -W\Delta y & &\text{weight } \mathbf{W} \text{ on particle} \\ &= -\left(\frac{1}{2}ks_2^2 - \frac{1}{2}ks_1^2\right) & &\text{spring stretched from } s_1 \text{ to } s_2 \end{aligned} \tag{52}$$

## 14.2 Principle of Work and Energy

**kinetic energy** ($T$): the energy in a particle due to its motion
**principle of work and energy**: the work done by a force on a particle is its change in energy

$$\begin{aligned} \Sigma U &= \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \\ T &= \frac{1}{2}mv^2 \\ T_1 + \Sigma U &= T_2 \end{aligned} \tag{53}$$

## 14.3 Principle of Work and Energy for a System of Particles

The internal forces in a rigid body do not cancel, so some work becomes heat. For a rigid body:

$$\Sigma T_1 + \Sigma U = \Sigma T_2 \tag{54}$$

Friction acts on many localized deformations, so its displacement $\Delta\mathbf{r}$ is less than expected.
An equal applied force thus does more work than friction, and this goes into internal energy.

## 14.4 Power and Efficiency

**power** ($P$): the amount of work done per unit time
**Watt** (W): the SI unit of power, $1\ \mathrm{W} = 1\ \mathrm{J} \cdot \mathrm{s}^2$
**horsepower** (hp): the FPS unit of power, $1\ \mathrm{hp} = 746\ \mathrm{W} = 550\ \mathrm{ft} \cdot \mathrm{lb} \cdot \mathrm{s}^{-1}$

$$P = \frac{\mathrm{d}U}{\mathrm{d}t} = \mathbf{F} \cdot \mathbf{v} \tag{55}$$

**efficiency** ($\epsilon$): the ratio of useful power output to power input

$$\epsilon = \frac{P_{out}}{P_{in}} = \frac{U_{out}}{U_{in}} < 1 \tag{56}$$

## 14.5 Conservative Forces and Potential Energy

**potential energy**: the energy in a particle due to its position
**potential function** ($V$): the algebraic sum of different potential energies
The gravitational $V_g$ and elastic $V_e$ potential energy of a particle are:

$$\begin{aligned} V_g &= Wy \\ V_e &= \frac{1}{2}ks^2 \\ V &= V_g + V_e \\ U &= V_1 - V_2 \end{aligned} \tag{57}$$

**gradient** ($\boldsymbol{\nabla}$): the operator that outputs a vector of all the partial derivatives of the input vector
**conservative force**: a force such that the work done by it is path-independent
Only the work done by a conservative force can be expressed as a difference in potential energy.
Weight and spring forces are conservative, but friction is not. Forces are only conservative if:

$$\mathbf{F} = -\boldsymbol{\nabla} V \tag{58}$$

### 14.6 Conservation of Energy

**conservation of energy**: if only conservative forces do work, then energy is conserved

$$\begin{aligned} T_1 + V_1 &= T_2 + V_2 \\ \Sigma\, T_1 + \Sigma\, V_1 &= \Sigma\, T_2 + \Sigma\, V_2 \end{aligned} \tag{59}$$

## 15 Kinetics of a Particle: Impulse and Momentum

### 15.1 Principle of Linear Impulse and Momentum

**linear momentum** ($\mathbf{L}$): the product of the mass and the velocity of a particle
**linear impulse** ($\mathbf{I}$): the integral of the resultant force with respect to time
**principle of linear impulse and momentum**: the linear impulse on a particle is its change in linear momentum

$$\begin{aligned} \mathbf{L} &= m\mathbf{v} \\ \mathbf{I} &= \int \mathbf{F}_R \ \mathrm{d}t \\ \mathbf{L}_1 + \mathbf{I} &= \mathbf{L}_2 \end{aligned} \tag{60}$$

**impulse and momentum diagrams**: free body diagrams of the impulse and momentum of a particle, rather than the forces acting on it

### 15.2 Principle of Linear Impulse for a System of Particles

Since internal forces cancel out, the linear impulse on a system of particles is the sum of the linear impulses of all external forces on the system. A system with total mass $m$ and centre of mass $\mathbf{r}_G$ is equivalent to a single particle with mass $m$ and velocity $\mathbf{v}_G$, at its centre of mass:

$$\begin{aligned} \Sigma\, \mathbf{L}_1 + \Sigma\, \mathbf{I} &= \Sigma\, \mathbf{L}_2 \\ m(\mathbf{v}_G)_1 + \Sigma\, \mathbf{I} &= m(\mathbf{v}_G)_2 \end{aligned} \tag{61}$$

### 15.3 Conservation of Linear Momentum for a System of Particles

**conservation of linear momentum**: if there are no external linear impulses on a system of particles, then linear momentum is conserved

$$\begin{aligned} \Sigma\, \mathbf{L}_1 &= \Sigma\, \mathbf{L}_2 \\ (\mathbf{v}_G)_1 &= (\mathbf{v}_G)_2 \end{aligned} \tag{62}$$

**impulsive forces**: forces that produce a significant impulse
**average impulsive force** ($\mathbf{F}_{avg}$): the impulse during some time, divided by that time

$$\mathbf{F}_{avg} = \frac{1}{\Delta t} \int \mathbf{F} \ \mathrm{d}t \tag{63}$$

**non-impulsive forces**: forces that produce a negligible impulse
Non-impulsive forces can be ignored when studying very short periods of time.

## 15.4 Impact

**impact**: the collision of two bodies during a very short period of time
**line of impact**: the line passing through the centres of mass of the impacting bodies
**central impact**: an impact such that both impacting bodies move along the line of impact
**oblique impact**: an impact such that an impacting body does not move along the line of impact
**deformation**: the process that pushes two impacting bodies together
**restitution**: the process that pushes two impacting bodies apart
Impacting bodies exert a deformation impulse $\int \mathbf{P}\ \mathrm{d}t$, move with the same velocity at maximum deformation, and exert a restitution impulse $\int \mathbf{R}\ \mathrm{d}t$. Both impulses act along the line of impact.
**coefficient of restitution** ($e$): the ratio of the restitution impulse to the deformation impulse

$$\begin{aligned} e &= \frac{\int \mathbf{R}\ \mathrm{d}t}{\int \mathbf{P}\ \mathrm{d}t} < 1 \\ &= \frac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1} \end{aligned} \tag{64}$$

**elastic impact**: an impact such that no kinetic energy is lost and $e = 1$
**plastic impact**: an impact such that both impacting bodies stick together and $e = 0$

## 15.5 Angular Momentum

**angular momentum** ($\mathbf{H}_O$): the moment of the linear momentum about point $O$
The direction of the angular momentum is the same as the direction of the corresponding moment:

$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v} \tag{65}$$

## 15.6 Relation Between Moment of a Force and Angular Momentum

The angular momentum is the integral of the resultant moment with respect to time:

$$\mathbf{H}_O = \int (M_R)_O\ \mathrm{d}t \tag{66}$$

## 15.7 Principle of Linear Impulse and Momentum

**angular impulse** ($\int (M_R)_O\ \mathrm{d}t$): the integral of the resultant moment with respect to time
**principle of angular impulse and momentum**: the angular impulse on a particle is its change in angular momentum

$$(\mathbf{H}_O)_1 + \int (M_R)_O\ \mathrm{d}t = (\mathbf{H}_O)_2 \tag{67}$$

**conservation of angular momentum**: if there are no external angular impulses on a system of particles, then angular momentum is conserved

$$\Sigma\,(\mathbf{H}_O)_1 = \Sigma\,(\mathbf{H}_O)_2 \tag{68}$$

**central force**: a force that acts at the centre of mass
Central forces can create linear impulses, but can never create angular impulses.