MECH260: Introduction to Mechanics of Materials
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Mechanics of Materials, 10th Edition; R. C. Hibbeler

1 Stress

1.1 Introduction

mechanics of materials: the study of the internal effects of stress and strain in a deformable solid body due to external loads

1.2 Equilibrium of a Deformable Body

load: a force or moment directly applied to a body
tension: describes a load that pulls on a body
compression: describes a load that pushes on a body
concentrated load: a load that acts on a single point
distributed load: a load that acts over a larger area
centroid ($(\bar{x}, \bar{y})$): the geometric centre of an area or volume
A concentrated load is statically equivalent to a single force on any point along its line of action.
A coplanar distributed load has a load profile $p(x, y)$ over an area A and is statically equivalent to a single resultant force F_R acting at the point on A under the centroid of the load profile:

$$\begin{aligned} F_R &= \int_A p(x,y)\ \mathrm{d}A \\ \bar{x} &= \frac{\int_A x\, p(x,y)\ \mathrm{d}A}{F_R} \\ \bar{y} &= \frac{\int_A y\, p(x,y)\ \mathrm{d}A}{F_R} \end{aligned} \tag{1}$$

support reaction: a reaction force caused by a support
A support produces a force if it prevents translation and a moment if it prevents rotation.
The necessary and sufficient equations of equilibrium are:

$$\Sigma\mathbf{F} = \mathbf{0} \qquad \Sigma\mathbf{M}_O = \mathbf{0} \tag{2}$$

statically determinate: the condition of a body whose reaction forces can be solved with only static equilibrium equations
statically indeterminate: the condition of a body that is not statically determinate
method of sections: a method used to find the resultant internal loads at a point, by cutting the body with a plane through that point, separating the two parts, drawing a free-body diagram of one part, and finally solving the equations of equilibrium
normal force ($\mathbf{N}$): the component of the resultant internal force that acts normal to the plane
shear force ($\mathbf{V}$): the component of the resultant internal force that acts tangent to the plane
torque ($\mathbf{T}$): the component of the internal moment that acts normal to the plane
bending moment ($\mathbf{M}$): the component of the internal moment that acts tangent to the plane
The resultant internal loads at the point O are $\mathbf{F}_R = \mathbf{N} + \mathbf{V}$ and $(M_R)_O = \mathbf{T} + \mathbf{M}$.
If all loads are coplanar, then $T = 0$, $N = \pm\Sigma F_x$, $V = \pm\Sigma F_y$, and $M = \pm\Sigma M_O$.
Sign conventions: positive $\mathbf{N}$ points outwards, positive $\mathbf{V}$ points 90 degrees clockwise from positive $\mathbf{N}$, and positive $\mathbf{M}$ tends to rotate the section from bottom to top.

1.3 Stress

An ideal solid body is **continuous** and **cohesive**: it consists of a uniform distribution of matter, which is all connected together, with no breaks or cracks. Everything in this course is ideal.
stress: the intensity of the internal force at a point
normal stress (σ): the component of the stress that acts normal to the plane
shear stress (τ): the component of the stress that acts tangent to the plane
If the internal force $\mathbf{F}$ is resolved into its components relative to the xy plane:

$$\sigma_z = \frac{\mathrm{d}F_z}{\mathrm{d}A} \qquad \tau_{zx} = \frac{\mathrm{d}F_x}{\mathrm{d}A} \qquad \tau_{zy} = \frac{\mathrm{d}F_y}{\mathrm{d}A} \tag{3}$$

state of stress: a diagram of the nine components of stress acting on a volume element

1.4 Average Normal Stress in an Axially Loaded Bar

An ideal material is **homogeneous** and **isotropic**: it has the same properties everywhere and in all directions. Wood is **anisotropic**, since it has different properties with or against the grain.
cross section: a section taken perpendicular to the longitudinal axis of a body
An axially loaded bar deforms uniformly, with a constant normal force N acting through the centroids of its cross sections, which have area A. The average normal stress is $\sigma_{avg} = N/A$.
bearing stress: compressive normal stress between two bodies
The bearing stress of a small pin or bolt can be calculated using its projected area.

1.5 Average Shear Stress

The average shear stress for an internal shear force V over a cross section of area A is $\tau_{avg} = V/A$. In reality, there are points where shear stress is much larger, but engineering codes allow the use of the above equation to analyze small elements. Double shear, with two sections of one body supporting the shear force, halves the average shear stress on the body.
complementary property of shear: for a volume element with shear stress τ_{zy} to remain in equilibrium, it must have three other shear stresses τ'_{zy}, τ_{yz}, and τ'_{yz}, on opposite sides, with opposite directions, but all with the same magnitude

1.6 Allowable Stress Design

The allowable load on each body must be less than its failure load to account for measurement errors, fabrication inaccuracies, unexpected loads, abrupt vibrations, atmospheric decay, and general variability in mechanical properties.
factor of safety (F.S.): the ratio of the failure load to the allowable load

$$\text{F.S.} = \frac{F_{fail}}{F_{allow}} = \frac{\sigma_{fail}}{\sigma_{allow}} = \frac{\tau_{fail}}{\tau_{allow}} \tag{4}$$

allowable stress design: design that is based on an allowable stress limit
If a body is subject to normal or shear force at a section, its required area at the section is:

$$A = \frac{N}{\sigma_{allow}} \qquad A = \frac{V}{\tau_{allow}} \tag{5}$$

2 Strain

2.1 Deformation

deformation: a change in the shape or size of a body due to a load

2.2 Strain

strain: the intensity of the deformation at a point
displacement (δ): the change in position due to a load
normal strain (ϵ): the relative change in length due to a load
The units for normal strain are usually in micrometres per metre, in inches per inch, or in percent.
Sign conventions: positive strain is tensile elongation; negative strain is compressive contraction.
A bar of original length L_0 that deforms to a length L and an infinitesimal line segment of original length Δs that deforms to a length $\Delta s'$ have normal strain:

$$\epsilon = \lim_{\Delta s \to 0} \frac{\Delta s' - \Delta s}{\Delta s} \qquad \epsilon_{avg} = \frac{L - L_0}{L_0} = \frac{\delta}{L} \tag{6}$$

shear strain (γ): the change in angle between two perpendicular line segments due to a load
Sign conventions: positive strain means smaller angle; negative strain means larger angle.
Two perpendicular line segments that deform to an angle of θ have shear strain:

$$\gamma = \frac{\pi}{2} - \theta = \arctan \frac{\delta}{L} \tag{7}$$

These strains deform a volume element with sides Δx, Δy, Δz into:

$$\begin{aligned} x &= (1+\epsilon_x)\Delta x & y &= (1+\epsilon_y)\Delta_y & z &= (1+\epsilon_z)\Delta_z \\ \theta_{xy} &= \frac{\pi}{2} - \gamma_{xy} & \theta_{yz} &= \frac{\pi}{2} - \gamma_{yz} & \theta_{xz} &= \frac{\pi}{2} - \gamma_{xz} \end{aligned} \tag{8}$$

state of strain: a diagram of the six components of strain acting on a volume element
small strain analysis: design that only allows small deformations; practical for engineering
In small strain analysis, $\sin\gamma \approx \gamma$, $\cos\gamma \approx 1$, and $\tan\gamma \approx \gamma$.

3 Mechanical Properties of Materials

3.1 The Tension and Compression Test

The strength of a material is intrinsic and must be found experimentally by various tests.

3.2 The Stress–Strain Diagram

In this chapter, unless otherwise specified, all stresses and strains are normal stresses and strains.
stress–strain diagram: an experimentally derived graph of stress versus strain
nominal stress or engineering stress: stress that uses the original cross-sectional area $\sigma = F/A_0$
nominal strain or engineering strain: strain that uses the original gauge length $\epsilon = \delta/L_0$
conventional stress–strain diagram: a stress–strain diagram using nominal stress and strain
elastic region: the first region in a stress–strain diagram, where the material exhibits elastic behaviour and can return to its original shape without permanent deformation
elastic limit: the largest stress for which the material can still return to its original shape

Hooke's law: in the elastic region, stress is directly proportional to strain

$$\sigma = E\epsilon \tag{9}$$

modulus of elasticity or Young's modulus (E): the proportionality constant between stress and strain in the elastic region
proportional limit (σ_{pl}): the largest stress for which Hooke's law applies
plastic region: the non-elastic region of a stress–strain diagram
yielding: the second region in a stress–strain diagram, where the material exhibits plastic behaviour and is permanently deformed, with strain increasing without any increase in stress
yield stress or yield point (σ_Y): the stress for which the material starts to permanently deform
strain hardening: the third region in a stress–strain diagram, where stress increases with strain
ultimate stress or ultimate strength (σ_u): the largest nominal stress before failure
necking: the last region in a stress–strain diagram, where the cross-sectional area begins to decrease in a localized region, creating a narrow neck that fails under lower nominal stress
fracture stress (σ_f): the nominal stress at failure
true stress: stress that uses the smallest instantaneous cross-sectional area $\sigma = F/A$
true strain: strain that uses the instantaneous gauge length $\epsilon = \delta/L$
true stress–strain diagram: a stress–strain diagram using true stress and strain, identical in the elastic and yielding regions, but with much higher stress during yielding and necking
true fracture stress: the true stress at failure

3.3 Stress–Strain Behaviour of Ductile and Brittle Materials

ductile material: any material that can undergo significant deformation before failure
Strain at failure or **percent elongation** and **percent reduction of area** quantify ductility.
If a body's length at failure is L_f and cross-sectional area of the neck at failure is A_f, then:

$$\begin{aligned} \text{percent elongation} &= \frac{L_f - L_0}{L_0}\frac{100\%}{1} = \epsilon\frac{100\%}{1} \\ \text{percent reduction of area} &= \frac{A_0 - A_f}{A_0}\frac{100\%}{1} = \frac{d_0^2 - d_f^2}{d_0^2}\frac{100\%}{1} \end{aligned} \tag{10}$$

Among ductile materials, soft steels, brass, molybdenum, and zinc yield without increasing stress. Most other metals, such as aluminum, go directly to strain hardening, with no clear yield point.
yield strength: the yield stress of a material with no clear yield point
The standard procedure to find the yield strength is with the **offset method**. If a line parallel to the beginning of the stress–strain curve is drawn from the strain axis at the desired strain, conventionally 0.2% for structural design, then the stress at their intersection is the yield strength. Assume that the elastic limit, proportional limit, yield point, and yield strength always coincide.
brittle material: any material that fails without significant deformation
Hard steels, cast iron, concrete, and plastics are brittle. Brittle materials have cracks that spread under tension and close under compression, so are much stronger in compression than tension. They suddenly fail in tension without yielding, and only have an average tensile fracture stress. Materials become more brittle at cold temperatures and more ductile at high temperatures. Brittle materials fail under normal stress, while ductile materials fail under shear stress.
stiffness: the ability of a material to resist deformation, quantified by the modulus of elasticity
residual strain or permanent set: the strain after a load is removed in the plastic region
The point at which the load is removed in the plastic region becomes the new yield point, and the body keeps the same modulus of elasticity. If the load is removed during strain hardening, then the yield point increases, but the size of the plastic region and thus ductility decreases.

3.4 Strain Energy

strain energy: the energy stored in a body due to deformation by an external load
strain energy density (u): the strain energy per unit volume of a material

$$u = \frac{\Delta U}{\Delta V} = \frac{1}{2}\sigma\epsilon \tag{11}$$

elastic strain energy density (u): the strain energy density of a material in the elastic region

$$u = \frac{1}{2}\frac{\sigma^2}{E} = \frac{1}{2}\epsilon^2 E \tag{12}$$

modulus of resilience (u_r): the elastic strain energy density at the proportional limit

$$u_r = \frac{1}{2}\sigma_{pl}\epsilon_{pl} = \frac{1}{2}\frac{\sigma_{pl}^2}{E} \tag{13}$$

modulus of toughness (u_t): the strain energy density at failure
The modulus of resilience is the maximum energy a material of unit volume can absorb before undergoing permanent deformation. The modulus of toughness is the total area under the stress–strain diagram, which is the maximum energy a material of unit volume can absorb before failure.

3.5 Poisson's Ratio

A body that, when subject to an axial force, elongates longitudinally, also contracts laterally, and vice versa. Within the elastic region of an ideal material, Poisson's ratio is constant.
Poisson's ratio (ν): the ratio of the lateral strain ϵ_{lat} to the longitudinal strain ϵ_{long}

$$0 \leq \nu = -\frac{\epsilon_{lat}}{\epsilon_{long}} \leq 0.5 \tag{14}$$

3.6 The Shear Stress–Strain Diagram

shear stress–strain diagram: an graph of shear stress versus shear strain
proportional limit (τ_{pl}): the largest shear stress for which Hooke's law applies
ultimate shear stress (τ_u): the largest nominal shear stress before failure
fracture stress (τ_f): the nominal shear stress at failure
modulus of rigidity or shear modulus of elasticity (G): the proportionality constant between shear stress and shear strain in the elastic region, used in Hooke's law for shear

$$\tau = G\gamma \tag{15}$$

The modulus of elasticity, modulus of rigidity, and Poisson's ratio are related by:

$$E = 2(1+\nu)G \tag{16}$$

3.7 Failure of Materials Due to Creep and Fatigue

creep: permanent deformation from stresses smaller than the yield stress under static loading, but which are applied for a long period of time or at very high temperatures
creep strength: the largest stress that does not exceed a given strain after a given time
fatigue: failure from stresses smaller than the yield stress, due to cyclic loading
fatigue strength or endurance limit: the largest stress below which a material will never fail
stress–cycle diagram: a graph of stress versus number of cycles to fatigue failure

4 Axial Load

4.1 Saint-Venant's Principle

Saint-Venant's principle: given two statically equivalent external loads on the same region, their effects are the same at points far enough away from the region
For an axially loaded bar, the largest dimension of its cross section is far enough away.

4.2 Elastic Deformation of an Axially Loaded Member

Given a bar of length L with internal normal force $N(x)$, cross-sectional area $A(x)$, and modulus of elasticity $E(x)$, in the elastic region, its total displacement δ is:

$$\delta = \int_0^L \frac{N(x)}{A(x)E(x)}\,\mathrm{d}x \tag{17}$$

Sign conventions: positive normal force and displacement cause tension and elongation; negative normal force and displacement cause compression and contraction.

4.3 Principle of Superposition

principle of superposition: the effects of a load divided into separate components is the algebraic sum of the effects of each component applied separately
The principle of superposition can be applied only if all loads are linearly related to their effects (physical linearity) and no loads significantly change the shape of the body (geometric linearity).

4.4 Statically Indeterminate Axially Loaded Members

compatibility condition: an equation that constrains displacement, due to physical geometry
load–displacement relationship: an equation that relates load to displacement, such as (17).
These conditions and relationships can be used for problems with statically indeterminate bodies.

4.5 The Force Method of Analysis for Axially Loaded Members

force method of analysis: choose one unknown load to be redundant and remove it so that the body becomes statically determinate, then equate the known displacement to the displacement caused by the known loads plus the displacement caused by the redundant load

4.6 Thermal Stress

Most materials linearly elongate when heated and linearly contract when cooled.
thermal strain (ϵ_T): strain created by temperature changes
linear coefficient of thermal expansion (α): the proportionality constant between thermal strain and change in temperature

$$\epsilon_T = \frac{\delta_T}{L} = \alpha \Delta T \tag{18}$$

thermal stress: stress created by thermal strain on a constrained body
Thermal displacement occurs in addition to any other displacements caused by other loads.

5 Torsion

5.1 Torsional Deformation of a Circular Shaft

When a torque is applied to a deformable circular shaft, all radial lines will remain straight but rotate and all cross sections will remain flat, while longitudinal lines tend to distort into a helix. Under torsion, brittle materials fracture in a plane 45 degrees to the axis due to normal stress, while ductile materials fracture in a plane normal to the axis due to shear stress.
angle of twist (ϕ): the angle by which a radial line rotates under torsion
The length and radius of a shaft do not change if the angle of twist is small.
As the angle of twist increases, a shear strain is induced. In a shaft of radius c, the shear strain varies linearly from 0 at its axis to γ_{max} at its outer surface. At a distance of ρ from its axis:

$$\gamma = \rho \frac{\mathrm{d}\phi}{\mathrm{d}x} = \left(\frac{\rho}{c}\right) \gamma_{max} \tag{19}$$

5.2 The Torsion Formula

second polar moment of area or polar moment of inertia (J): the ability of a body to resist torsion, based purely on the geometry of its cross sections

$$J = \int_A \rho^2 \ \mathrm{d}A \qquad J_{circle} = \frac{\pi c^4}{2} \tag{20}$$

Given a shaft of radius c, internal torque T, and second polar moment of area J, in the elastic region, its shear stress τ at a distance ρ from its longitudinal axis is:

$$\tau = \frac{T\rho}{J} = \left(\frac{\rho}{c}\right) \tau_{max} \tag{21}$$

The outer surfaces of any free body are always free of both normal and shear stresses.

5.3 Power Transmission

The power P transmitted by a rotating shaft is the applied torque T times angular velocity ω:

$$P = T\omega \tag{22}$$

5.4 Angle of Twist

Given a shaft of length L with internal torque $T(x)$, second polar moment of area $J(x)$, and modulus of rigidity $G(x)$, in the elastic region, its angle of twist ϕ is:

$$\phi = \int_0^L \frac{T(x)}{J(x)G(x)} \ \mathrm{d}x \tag{23}$$

Sign conventions: positive torque points out from the shaft. By the right-hand rule, if you point your right thumb in the direction of a torque, then your fingers curl in the direction that it twists.

5.5 Statically Indeterminate Torque-Loaded Members

(23) is a load–displacement relationship, which can be used for statically indeterminate bodies.

6 Bending

6.1 Shear and Moment Diagrams

Beams support loads perpendicular to their longitudinal axis. A **cantilevered** beam is fixed at one end and free at the other, an **overhanging** beam has at least one end hanging over a support, and a **simply supported** beam is pinned at one end and roller supported at the other.
shear diagram: a graph of shear stress versus position along a beam
moment diagram: a graph of bending moment versus position along a beam
The shear and moment diagrams are discontinuous, or their slopes are discontinuous, where external forces and moments are applied, so both $V(x)$ and $M(x)$ must be analyzed piecewise.

6.2 Graphical Method for Constructing Shear and Moment Diagrams

Sign conventions: under bending, positive forces act upwards, positive moments bend clockwise.
At a position with only a distributed force $w(x)$, the shear and moment diagrams are related by:

$$\frac{\mathrm{d}V}{\mathrm{d}x} = w \qquad \frac{\mathrm{d}M}{\mathrm{d}x} = V \qquad \Delta V = \int w \ \mathrm{d}x \qquad \Delta M = \int V \ \mathrm{d}x \tag{24}$$

At a position with a concentrated force F or concentrated bending moment M_0:

$$\Delta V = F \qquad \Delta M = M_0 \tag{25}$$

6.3 Bending Deformation of a Straight Member

When a bending moment is applied to a deformable symmetric beam, all lateral lines will remain straight but rotate and all cross sections will remain flat, while longitudinal lines tend to bend.
neutral surface: the surface on which all longitudinal lines neither elongate nor contract
The beam is in tension on one side of the neutral surface and in compression on the other side.
neutral axis: the line of intersection between the neutral surface and the cross section
The neutral axis is the horizontal line that intersects the centroid of the cross section.
In any beam where the maximum distance from the neutral axis is c, the normal strain always varies linearly from 0 on the neutral axis to ϵ_{max} at c. At a distance of y from the neutral axis:

$$\epsilon = -\left(\frac{y}{c}\right)\epsilon_{max} \tag{26}$$

6.4 The Flexure Formula

second planar moment of area or area moment of inertia (I): the ability of a body to resist bending, based purely on the geometry of its cross sections

$$I = \int_A y^2 \ \mathrm{d}A \qquad I_{rectangle} = \frac{1}{12}bh^3 \tag{27}$$

J and I are both second moments of area, but J is calculated with respect to an axis normal to the cross section, while I is calculated with respect to an axis in the plane of the cross section.
I is always calculated with respect to the neutral axis, unless otherwise specified.
Given a beam with maximum distance c, internal bending moment M, and second planar moment of area I, in the elastic region, its normal stress σ at a distance y from the neutral axis is:

$$\sigma = -\frac{My}{I} = -\left(\frac{y}{c}\right)\sigma_{max} \tag{28}$$

7 Transverse Shear

7.1 Shear in Straight Members

longitudinal shear: shear that acts parallel to the longitudinal axis
transverse shear: shear that acts perpendicular to the longitudinal axis
The complementary property of shear creates longitudinal shear from transverse shear.
Shear stresses create shear strains that warp cross sections, but negligibly for slender beams.

7.2 The Shear Formula

first moment of area (Q): the ability of a body to resist shear, based purely on geometry
The first moment of area depends on the area A and centroid $(\bar{x}, \bar{y})$ of the cross section:

$$Q = \int_A y \, \mathrm{d}A = \bar{y}A \qquad \bar{y} = \frac{\int_A y \, \mathrm{d}A}{\int_A \mathrm{d}A} = \frac{Q}{A} \tag{29}$$

Given a beam with internal shear force V and second moment of area I, in the elastic region, its shear stress τ over a longitudinal plane intersecting the cross section at a line of width t, such that the first moment of area, about the neutral axis, of the cross section above this line is Q, is:

$$\tau = \frac{VQ}{It} \tag{30}$$

The shear formula cannot be used at stress concentrations, nor in short and flat cross sections, where the maximum shear stress is much larger than the average shear stress. The shear formula can only be used on lines that pass perpendicular to the outer surface, which is free of stress.

12 Deflection of Beams and Shafts

12.1 The Elastic Curve

elastic curve: the line through the centroids of the cross sections of a beam
flexural rigidity (EI): the ability of a body to resist deflection
Given a beam with internal moment M, modulus of elasticity E, and second planar moment of area I, in the elastic region, its flexural rigidity is EI and its curvature $1/\rho$ is:

$$\frac{1}{\rho} = \frac{M}{EI} \tag{31}$$

Energy methods are needed to find longitudinal deflections, which are usually negligible.

12.2 Slope and Displacement by Integration

The deflection on the elastic curve is given by $v(x)$. Most engineering codes restrict the maximum deflection, so the slope $v'(x)$ is so small that $v''(x) = 1/\rho$. Using (24) and taking EI as constant:

$$EI\frac{\mathrm{d}^2x}{\mathrm{d}x^2} = M(x) \qquad EI\frac{\mathrm{d}^3x}{\mathrm{d}x^3} = V(x) \qquad EI\frac{\mathrm{d}^4x}{\mathrm{d}x^4} = w(x) \tag{32}$$

Sign conventions: positive deflection is upwards. If x is positive to the right, then positive slope is measured counterclockwise from the x axis. The constants of integration come from boundary and continuity conditions: the deflection at supports that resist forces is fixed, the slope at supports that resist moments is fixed, and both the deflection and slope must be continuous.