MATH257: Partial Differential Equations
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4 Fourier Series and PDEs

4.1 Boundary Value Problems

differential equation: an equation of an unknown function and its derivatives
initial condition: a concrete value of the function or its derivatives at a given time
boundary condition: a concrete value of the function or its derivatives at a given location
side condition: an initial condition or a boundary condition
boundary value problem: a differential equation with boundary conditions
Consider the boundary value problems:

$$\begin{aligned}
&\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \lambda u = 0 \quad x \in [a,b] \quad u(a) = 0 \quad u(b) = 0 \\
&\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \lambda u = 0 \quad x \in [a,b] \quad \frac{\mathrm{d}u(a)}{\mathrm{d}x} = 0 \quad \frac{\mathrm{d}u(b)}{\mathrm{d}x} = 0 \\
&\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \lambda u = 0 \quad x \in [a,b] \quad u(a) = u(b) \quad \frac{\mathrm{d}u(a)}{\mathrm{d}x} = \frac{\mathrm{d}u(b)}{\mathrm{d}x}
\end{aligned} \tag{1}$$

eigenvalue: a concrete value of λ such that a nonzero solution exists
eigenfunction: the nonzero solution that corresponds to some eigenvalue
The eigenvalues and eigenfunctions of (1) are respectively:

$$\begin{aligned}
&\lambda_n = \left(\frac{n\pi}{b-a}\right)^2 \quad u_n = b_n \sin\left(\frac{n\pi}{b-a}(x-a)\right) && n = 1, 2, 3, \ldots \\
&\lambda_n = \left(\frac{n\pi}{b-a}\right)^2 \quad u_n = a_n \cos\left(\frac{n\pi}{b-a}(x-a)\right) && n = 0, 1, 2, \ldots \\
&\lambda_n = \left(\frac{2n\pi}{b-a}\right)^2 \quad u_n = a_n \cos\left(\frac{2n\pi}{b-a}(x-a)\right) + b_n \sin\left(\frac{2n\pi}{b-a}(x-a)\right) && n = 0, 1, 2, \ldots
\end{aligned} \tag{2}$$

inner product: for two functions f, g with respect to a weight function r, the integral

$$\langle f, g \rangle = \int_a^b f(x)g(x)r(x)\ \mathrm{d}x \tag{3}$$

orthogonality: the property of two functions f, g where $\langle f, g \rangle = 0$
All eigenfunctions of (1) that correspond to different eigenvalues (2) are orthogonal if $r(x) = 1$.
Fredholm alternative: either $u(y, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}) = 0$ has a nonzero solution, or $u(y, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}) = 0$ has a unique solution for all $f(x)$ continuous on $[a, b]$
The Fredholm alternative applies to (1).

4.2 The Trigonometric Series

periodic: describes a function f with period P such that $f(x) = f(x + P)$ for all x
periodic extension: the P-periodic function F such that $F(x) = f(x)$ where f is defined
Consider the third equation in (1), with $a = -L$ and $b = L$:

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \lambda u = 0 \qquad x \in [-L, L] \quad u(-L) = u(L) \quad \frac{\mathrm{d}u(-L)}{\mathrm{d}x} = \frac{\mathrm{d}u(L)}{\mathrm{d}x} \tag{4}$$

Fourier series: the decomposition of a $2L$-periodic function into the eigenfunctions of (4)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \tag{5}$$

Consider (3), with $a = -L$, $b = L$, and $r(x) = 1$. Then the coefficients of the Fourier series are:

$$\begin{aligned}
\langle f, g\rangle &= \int_{-L}^{L} f(x)g(x)\ \mathrm{d}x \\
a_0 &= 2\frac{\langle f(x), 1\rangle}{\langle 1, 1\rangle} &&= \frac{1}{L}\int_{-L}^{L} f(x)\ \mathrm{d}x \\
a_n &= \frac{\langle f(x), \cos\left(\frac{n\pi}{L}x\right)\rangle}{\langle \cos\left(\frac{n\pi}{L}x\right), \cos\left(\frac{n\pi}{L}x\right)\rangle} &&= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)\ \mathrm{d}x \\
b_n &= \frac{\langle f(x), \sin\left(\frac{n\pi}{L}x\right)\rangle}{\langle \sin\left(\frac{n\pi}{L}x\right), \sin\left(\frac{n\pi}{L}x\right)\rangle} &&= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)\ \mathrm{d}x
\end{aligned} \tag{6}$$

Gibbs phenomenon: as more terms in the Fourier series are taken, the error width near a discontinuity decreases, but the error magnitude does not

4.3 More on the Fourier Series

piecewise continuous: describes a function where a finite set of points exists such that the function is continuous between these points and the one-sided limit is finite at these points
piecewise smooth: describes a function that is differentiable everywhere except a finite set of points, and whose derivative is piecewise continuous
The Fourier series of a piecewise smooth function converges everywhere to the average of the one-sided limits at each point. The derivative of a continuous piecewise smooth function with a piecewise smooth derivative is the derivative of its Fourier series. The antiderivative of a piecewise smooth function is the antiderivative of its Fourier series.
The convergence of the Fourier coefficients for a polynomial function is $1/n^3$ for a differentiable function, $1/n^2$ for a merely continuous function, and $1/n$ for a discontinuous function.

4.4 Sine and Cosine Series

odd: describes a function f such that $f(-x) = -f(x)$
even: describes a function f such that $f(-x) = f(x)$
odd periodic extension: the odd $2L$-periodic extension of a function defined on $[0, L]$
even periodic extension: the even $2L$-periodic extension of a function defined on $[0, L]$

$$F_{odd}(x) = \begin{cases} f(x) & 0 \leq x \leq L \\ -f(-x) & -L < x < 0 \end{cases}$$
$$F_{even}(x) = \begin{cases} f(x) & 0 \leq x \leq L \\ f(-x) & -L < x < 0 \end{cases} \tag{7}$$

Fourier sine series: the Fourier series of the odd periodic extension, with only sine terms
Fourier cosine series: the Fourier series of the even periodic extension, with only cosine terms

4.5 Applications of Fourier Series

A variety of problems can be solved by rewriting piecewise smooth functions as Fourier series. Consider a mass m on a spring, with spring constant k, external force F, and coefficient of friction c, which has the following second order linear ODE:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + c\frac{\mathrm{d}x}{\mathrm{d}t} + kx = F(t) \tag{8}$$

resonance: describes a mass-spring system with a solution whose amplitude increases over time
Resonance occurs when there is no friction and terms in the external force coincide with terms in the solution to the homogeneous problem.

4.6 PDEs, Separation of Variables, and the Heat Equation

partial differential equation (PDE): a differential equation of a multivariate function
linear PDE: a PDE that is linear in the function and its derivatives
linear homogeneous PDE: a linear PDE whose constant term is zero
homogeneous side condition: a side condition where the function or its derivatives is zero
superposition: any linear combination of solutions of a linear homogeneous PDE with homogeneous side conditions is also a solution
one-dimensional heat equation: the linear PDE that models temperature $u(x, t)$ as a function of time t and the length x along a thin insulated rod with thermal conductivity $k > 0$

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad x \in [0, L] \quad u(0, t) = 0 \quad u(L, t) = 0 \quad u(x, 0) = f(x) \tag{9}$$

The one-dimensional heat equation can be solved by separation of variables and superposition:

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ X\frac{\mathrm{d}T}{\mathrm{d}t} &= k\frac{\mathrm{d}^2X}{\mathrm{d}x^2}T \\ \frac{1}{kT}\frac{\mathrm{d}T}{\mathrm{d}t} &= \frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2} = -\lambda \end{aligned} \tag{10}$$

Note that two functions of different variables can only equal if they are both constant, and for a nonzero solution, $T(t) \neq 0$ for the side conditions:

$$\begin{gathered} u(0,t) = X(0)T(t) = 0 \to X(0) = 0 \\ u(L,t) = X(L)T(t) = 0 \to X(L) = 0 \\ \frac{\mathrm{d}^2X}{\mathrm{d}x^2} + \lambda X = 0 \end{gathered} \tag{11}$$

This is (1), whose eigenvalues λ_n and eigenfunctions $X_n(x)$ depend on the boundary conditions:

$$\begin{aligned} \frac{\mathrm{d}T}{\mathrm{d}t} + \lambda kT = 0 \\ T_n(t) = e^{-\lambda_n kt} \\ u(x,t) = \sum_{n=0}^{\infty} c_n X_n(x) e^{-\lambda_n kt} \end{aligned} \tag{12}$$

The constants c_n are given by rewriting the initial conditions as a Fourier series:

$$u(x,0) = \sum_{n=0}^{\infty} c_n X_n(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) = f(x) \tag{13}$$

Dirichlet conditions: the boundary conditions that model ends fixed at zero temperature

$$X(0) = 0 \qquad X(L) = 0 \tag{14}$$

Neumann conditions: the boundary conditions that model ends insulated from heat

$$X'(0) = 0 \qquad X'(L) = 0 \tag{15}$$

Robin conditions: the boundary conditions that model ends immersed in some medium

$$hX(0) - X'(0) = 0 \qquad hX(L) + X'(L) = 0 \tag{16}$$

The solution to the heat equation is infinitely differentiable at all $t > 0$, which instantly smooths out the initial condition.

4.7 One Dimensional Wave Equation

one-dimensional wave equation: the linear PDE that models displacement $u(x,t)$ as a function of time t and the length x along a tensioned string with propagation speed $c > 0$

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad x \in [0,L] \quad & u(0,t) = 0 \qquad\quad u(L,t) = 0 \\ & u(x,0) = f(x) \quad \frac{\partial u(x,0)}{\partial t} = g(x) \end{aligned} \tag{17}$$

The one-dimensional wave equation can be solved by adding the solutions to two almost homogeneous wave equations w and z identical to u except in their initial conditions:

$$\begin{aligned} u(x,t) &= w(x,t) + z(x,t) \\ w(x,0) &= 0 \qquad\qquad z(x,0) = f(x) \\ \frac{\partial w(x,0)}{\partial t} &= g(x) \qquad \frac{\partial z(x,0)}{\partial t} = 0 \end{aligned} \tag{18}$$

The almost homogeneous wave equations can be solved by separation of variables:

$$\begin{aligned} w(x,t) &= X(x)T(t) \\ X\frac{\mathrm{d}^2T}{\mathrm{d}t^2} &= c^2\frac{\mathrm{d}^2X}{\mathrm{d}x^2}T \\ \frac{1}{c^2T}\frac{\mathrm{d}^2T}{\mathrm{d}t^2} &= \frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2} = -\lambda \\ \frac{\mathrm{d}^2X}{\mathrm{d}x^2} + \lambda X &= 0 \\ \frac{\mathrm{d}^2T}{\mathrm{d}t^2} + \lambda c^2 T &= 0 \\ T_n(t) &= A_n\cos\left(c\sqrt{\lambda}_n t\right) + B_n\sin\left(c\sqrt{\lambda}_n t\right) \end{aligned} \tag{19}$$

Note that for a nonzero solution, $X(x) \neq 0$ for the boundary conditions. The constants c_n are given by rewriting the nonhomogeneous initial condition as a Fourier series:

$$\begin{aligned} w(x,0) &= X(x)T(0) = 0 \to T(0) = 0 \to A_n = 0 \\ w(x,t) &= \sum_{n=0}^{\infty} c_n X_n(x)\sin\left(c\sqrt{\lambda}_n t\right) \\ \frac{\partial w(x,t)}{\partial t} &= \sum_{n=0}^{\infty} c\sqrt{\lambda}_n c_n X_n(x)\cos\left(c\sqrt{\lambda}_n t\right) \\ \frac{\partial w(x,0)}{\partial t} &= \sum_{n=0}^{\infty} c\sqrt{\lambda}_n c_n X_n(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{L}x\right) + b_n\sin\left(\frac{n\pi}{L}x\right) = g(x) \end{aligned} \tag{20}$$

The same principles apply to the other almost homogeneous wave equation. The solution to the original wave equation is their sum.

4.8 D'Alembert Solution of the Wave Equation

By the change of variables $\xi = x - at$ and $\eta = x + at$:

$$u(\xi,\eta) = A(\xi) + B(\eta) = u(x,t) = A(x-at) + B(x+at) \tag{21}$$

characteristic coordinates: the coordinates ξ and η used to solve the wave equation
d'Alembert's formula: the solution to the wave equation using the characteristic coordinates
Given that $F(x)$ and $G(x)$ are the odd periodic extensions of $f(x)$ and $g(x)$ with Dirichlet conditions, or the even periodic extensions of $f(x)$ and $g(x)$ with Neumann conditions:

$$u(x,t) = \frac{F(x-at) + F(x+at)}{2} + \frac{1}{2c}\int_{x-at}^{x+at} G(s)\ \mathrm{d}s \tag{22}$$

4.9 Steady State Temperature and the Laplacian

Laplacian (Δ): the divergence of the gradient
Laplace equation: the linear PDE that models steady state temperature u as function of space

$$\begin{aligned} \Delta u &= 0 \\ \Delta u(x,y) &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad x \in [0,W] \qquad y \in [0,H] \\ u(x,H) &= f_n(x) \quad u(W,y) = f_e(y) \quad u(x,0) = f_s(x) \quad u(0,y) = f_w(y) \end{aligned} \tag{23}$$

harmonic function: a solution to the Laplace equation
The two-dimensional Laplace equation can be solved by adding the solutions to four almost homogeneous Laplace equations n, e, w, and s identical to u except in their initial conditions:

$$\begin{aligned} u(x,y) = n(x,y) \quad & + e(x,y) & + s(x,y) & + w(x,y) \\ n(x,H) = f_n(x) \quad & e(x,H) = 0 & s(x,H) = 0 & \quad w(x,H) = 0 \\ n(W,y) = 0 \quad & e(W,y) = f_e(y) & s(W,y) = 0 & \quad w(W,y) = 0 \\ n(x,0) = 0 \quad & e(x,0) = 0 & s(x,0) = f_s(x) & \quad w(x,0) = 0 \\ n(0,y) = 0 \quad & e(0,y) = 0 & s(0,y) = 0 & \quad w(0,y) = f_w(y) \end{aligned} \tag{24}$$

The almost homogeneous Laplace equations can be solved by separation of variables:

$$\begin{aligned} s(x,t) &= X(x)T(t) \\ \frac{\mathrm{d}^2 X}{\mathrm{d}x^2} Y + X \frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} &= 0 \\ \frac{-1}{X}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} &= \frac{1}{Y}\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} = \lambda \\ \frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + \lambda X &= 0 \\ \frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} - \lambda Y &= 0 \\ Y_n(y) &= A_n \cosh\left(\sqrt{\lambda}_n y\right) + B_n \sinh\left(\sqrt{\lambda}_n y\right) \end{aligned} \tag{25}$$

The constants c_n are given by rewriting the nonhomogeneous initial condition as a Fourier series:

$$\begin{aligned} s(x,H) &= X(x)T(h) = 0 \to T(h) = 0 \to B_n = -A_n \frac{\cosh\left(\sqrt{\lambda}_n h\right)}{\sinh\left(\sqrt{\lambda}_n h\right)} \\ Y_n(y) &= A_n \left(\frac{\sinh\left(\sqrt{\lambda}_n (h-y)\right)}{\sinh\left(\sqrt{\lambda}_n h\right)} \right) \\ s(x,t) &= \sum_{n=0}^{\infty} c_n X_n(x) \left(\frac{\sinh\left(\sqrt{\lambda}_n (h-y)\right)}{\sinh\left(\sqrt{\lambda}_n h\right)} \right) \\ s(x,0) &= \sum_{n=0}^{\infty} c_n X_n(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) = f_s(x) \end{aligned} \tag{26}$$

The same principles apply to the other almost homogeneous Laplace equations. The solution to the original Laplace equation is their sum.

4.10 Dirichlet Problem in the Circle and the Poisson Kernel

The Laplace equation in polar coordinates is:

$$\begin{aligned} \Delta u(r,\theta) = \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} = 0 \qquad r \in [r_i, r_f] \qquad \theta \in [\theta_i, \theta_f] \\ u(r_i,\theta) = f(\theta) \qquad u(r_f,\theta) = g(\theta) \end{aligned} \tag{27}$$

The Laplace equation in polar coordinates can be solved by separation of variables:

$$\begin{aligned} u(r,\theta) &= R(r)\Theta(\theta) \\ \frac{1}{r^2}R\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2} + \frac{1}{r}\frac{\mathrm{d}R}{\mathrm{d}r}\Theta + \frac{\partial^2 R}{\partial r^2}\Theta &= 0 \\ \frac{1}{\Theta}\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2} = \frac{-r}{R}\frac{\mathrm{d}R}{\mathrm{d}r} - \frac{r^2}{R}\frac{\mathrm{d}^2R}{\mathrm{d}r^2} &= -\lambda \\ \frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2} + \lambda\Theta &= 0 \\ r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2} + r\frac{\mathrm{d}R}{\mathrm{d}r} - \lambda R &= 0 \end{aligned} \tag{28}$$

Since Θ is 2π-periodic, $\lambda = n^2$. Guess $R = r^s$ or $R = r^s \ln r$. If $r_i = 0$, then an implied boundary condition is that $u(r_i,\theta)$ is finite, so $B = 0$ and set $A = 1$:

$$\begin{aligned} \Theta_0 &= \frac{a_0}{2} \\ \Theta_n &= a_n\cos(n\theta) + b_n\sin(n\theta) \\ R_0 &= A + B\ln r = 1 \\ R_n &= Ar^n + Br^{-n} = r^n \end{aligned} \tag{29}$$

The constants c_n are given by rewriting the initial condition as a Fourier series:

$$\begin{aligned} u(r,\theta) &= \frac{a_0}{2} + \sum_{n=0}^{\infty} r^n\Theta_n(\theta) \\ u(r_f,\theta) &= \frac{a_0}{2} + \sum_{n=0}^{\infty} r_f^n\Theta_n(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\theta) + b_n\sin(n\theta) = g(\theta) \end{aligned} \tag{30}$$

The same principles apply to the other almost homogeneous Laplace equations. The solution to the original Laplace equation is their sum.
Poisson kernel (P): a function that helps solve Laplace equations numerically

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta,\alpha)g(\alpha)\ \mathrm{d}\alpha \tag{31}$$

The solution to the Laplace equation in polar coordinates with Dirichlet conditions using the Poisson kernel is:

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta-\alpha)+r^2}g(\alpha)\ \mathrm{d}\alpha \tag{32}$$

5 Eigenvalue Problems

5.1 Sturm-Liouville Problems

Sturm-Liouville problem: an ODE with the following form and boundary conditions

$$\begin{aligned} (py')' - qy + \lambda ry &= 0 & x \in (a,b) \\ \alpha_1 y(a) - \alpha_2 y'(a) &= 0 & \alpha_1^2 + \alpha_2^2 > 0 \\ \beta_1 y(b) + \beta_2 y'(b) &= 0 & \beta_1^2 + \beta_2^2 > 0 \end{aligned} \tag{33}$$

All second-order linear ODEs $-P(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - Q(x)\frac{\mathrm{d}y}{\mathrm{d}x} + Ry = \lambda Y$ can be written in the form (33):

$$\begin{aligned} \mu(x) &= \frac{1}{P} e^{\int_a^x \frac{Q(s)}{P(s)}\ \mathrm{d}s} \\ p(x) &= P(x)\mu(x) \\ q(x) &= R(x)\mu(x) \\ r(x) &= \mu(x) \end{aligned} \tag{34}$$

regular Sturm-Liouville problem: a Sturm-Liouville problem such that $p(x)$, $p'(x)$, $q(x)$, $r(x)$ are continuous on $[a,b]$, and $p(x) > 0$, $q(x) \geq 0$, $r(x) > 0$ on $[a,b]$, and α_1, α_2, β_1, $\beta_2 \geq 0$
For a regular Sturm-Liouville problem, all distinct eigenfunctions y_m, y_n are orthogonal with respect to its weight function $r(x)$:

$$\int_a^b y_m(x) y_n(x) r(x)\ \mathrm{d}x = 0 \tag{35}$$

For a regular Sturm-Liouville problem, the Fredholm alternative applies, and:

$$\begin{aligned} &0 \leq \lambda_1 < \lambda_2 < \cdots \\ &\lim_{n\to\infty} \lambda_n = \infty \end{aligned} \tag{36}$$

The decomposition of a piecewise smooth continuous function into the eigenfunctions of (33) is:

$$\begin{aligned} c_n &= \frac{\langle x, y_n(x)\rangle}{\langle y_n(x), y_n(x)\rangle} = \frac{\int_a^b f(x) y_n(x) r(x)\ \mathrm{d}x}{\int_a^b y_n^2(x) r(x)\ \mathrm{d}x} \\ f(x) &= \sum_{n=1}^{\infty} c_n y_n(x) \end{aligned} \tag{37}$$

5.2 Application of Eigenfunction Series

Regular Sturm-Liouville eigenvalue decompositions can solve problems.

5.3 Steady Periodic Solutions

steady periodic solution: a particular solution to a nonhomogeneous PDE whose amplitude remains constant
The solution to the homogeneous heat equation decays over time, so its steady periodic solutions are more important. All solutions to the wave equation are steady periodic solutions.

7 Power Series Methods

7.1 Power Series

power series: an infinite series involving powers of one variable

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k \tag{38}$$

partial sum (S_n): the sum of the first finite number of terms of a series

$$S_n(x) = \sum_{k=0}^{n} a_k (x - x_0)^k \tag{39}$$

divergent series: a series whose limit $\lim_{n\to\infty} S_n(x)$ only exists at x_0
convergent series: a series whose limit $\lim_{n\to\infty} S_n(x)$ exists at some point $x \neq x_0$
absolutely convergent series: a convergent series whose absolute limit $\lim_{n\to\infty} \sum_{k=0}^{n} |a_k||(x - x_0)^k|$ exists at some point $x \neq x_0$
radius of convergence (ρ): the number such that a series converges absolutely when $x_0 - \rho < x < x_0 + \rho$ and diverges when $x_0 - \rho > x$ or $x > x_0 + \rho$
ratio test: if $L = \lim_{n\to\infty} |\frac{a_{k+1}}{a_k}|$ exists, then it converges absolutely if $L < 1$, diverges if $L > 1$
analytic function: a function that can be represented by a power series
Taylor series: a power series involving derivatives equal to all analytic functions

$$\sum_{k=0}^{\infty} \frac{1}{k!} \frac{\mathrm{d}^k f(x_0)}{\mathrm{d}x^k} (x - x_0)^k \tag{40}$$

Power series can be added, subtracted, multiplied, divided, differentiated, and integrated term by term within their radii of convergence.

7.2 Series Solutions of Linear Second Order ODEs

Consider the linear second order homogeneous ODE with polynomial coefficients:

$$p(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + q(x)\frac{\mathrm{d}y}{\mathrm{d}x} + r(x)y = 0 \tag{41}$$

Solve this ODE by substituting the power series:

$$y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k \tag{42}$$

ordinary point: the point x_0 iff $p(x_0) \neq 0$, so that $q(x)/p(x)$ and $r(x)/p(x)$ are defined
singular point: the point x_0 iff $p(x_0) = 0$, so that $q(x)/p(x)$ and $r(x)/p(x)$ are undefined
regular singular point: the singular point x_0, iff both limits $\lim_{x\to x_0}(x - x_0)q(x)/p(x)$ and $\lim_{x\to x_0}(x - x_0)^2 r(x)/p(x)$ are finite
recurrence relation: an equation that relates arbitrary coefficients of a power series

7.3 Singular Points and the Method of Frobenius

method of Frobenius: if (41) has a regular singular point at x_0, then the following power series is a solution

$$y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^{k+r} \tag{43}$$

indical equation: the equation between the first coefficient and zero
Let a_0 be the first non-zero term in the power series. To satisfy (41), all terms, including the first term, must be zero, so consider the roots of the indical equation:

$$\begin{aligned}
&y_1(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^{k+r_1} && y_2(x) = \sum_{k=0}^{\infty} b_k (x - x_0)^{k+r_2} \\
&\qquad \text{two real roots } r_1,\ r_2,\ r_1 - r_2 \notin \mathbb{Z} \\
&y_1(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^{k+r_1} && y_2(x) = \sum_{k=0}^{\infty} b_k (x - x_0)^{k+r_2} + C y_1 \ln x \\
&\qquad \text{two real roots } r_1,\ r_2,\ r_1 - r_2 \in \mathbb{Z} \\
&y_1(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^{k+r} && y_2(x) = \sum_{k=0}^{\infty} b_k (x - x_0)^{k+r} + y_1 \ln x \\
&\qquad \text{one doubled root } r \\
&y_1(x) = \Re \sum_{k=0}^{\infty} a_k (x - x_0)^{k+r_1} && y_2(x) = \Im \sum_{k=0}^{\infty} a_k (x - x_0)^{k+r_1} \\
&\qquad \text{two complex roots } r_1,\ r_2
\end{aligned} \tag{44}$$

Bessel's equation: the linear second order homogeneous ODE

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x \frac{\mathrm{d}y}{\mathrm{d}x} + \left(x^2 - p^2\right) y = 0 \tag{45}$$

Bessel's equation can be solved using the method of Frobenius.